Group classification for the nonlinear heat conductivity equation

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Abstract

Symmetry properties of the nonlinear heat conductivity equations of the general form $u_t = [E(x,u)u_x]_x + H(x,u)$ is studied. The point symmetry analysis of these equations is considered as well as an equivalence classification which admits an extension by one dimension of the principal Lie algebra of the equation. The invariant solutions of equivalence transformations and classification of the nonlinear heat conductivity equations among with additional operators are also given.

1 Introduction

Symmetry properties of mathematical models of heat conductivity and diffusion processes [8] are traditionally formulated in terms of nonlinear differential equations which often envisage us with difficulties in studying. To solve this problem, symmetry methods play a key role for finding their exact solutions, similar solutions and invariants [2, 4, 16, 19, 20].

In this study, we generalize the study of a class of the nonlinear heat conductivity equations (HCEs) which has been recently studied in some special cases [13, 19, 20]. We deal with the class of nonlinear heat conductivity equations of the general form

$$\text{HCE}: \quad u_t = [E(x,u)u_x]_x + H(x,u), \tag{1.1}$$

in which we assumed that E, H are sufficiently smooth functions, $E_x, E_u, H_x, H_u \neq 0$, u is treated as the dimensionless temperature, t and x are the dimensionless time and space variables and E is the thermal conductivity.

The Lie point symmetry in linear and nonlinear special cases of our problem was investigated. For instance the problem for the case in which E just depends to u and $E_u = 0$ corresponding to the linear case, the case where the class of nonlinear one-dimensional diffusion equations when $E = E(u), H = 0$, the class of diffusion-reaction equations when $E = E(u), H = \texttt{const}$, the case which the thermal conductivity is a power function of the temperature and additional equivalence transformations, conditional equivalence groups and nonclassical symmetries have all investigated and listed in Table 1 of [20]. The point symmetry group of nonlinear fin equations of class (1.1) were considered in a number of papers. Moreover in [13], preliminary group classification of nonlinear fin equation was studied in the general form.

Class of Eqs. (1.1) generalizes a great number of the known nonlinear second order equations describing various processes in biology, ecology, physics and chemistry (see [2] and also [3] with references therein). Nonlinear heat equations in one or higher dimensions are also studied in literature by using both symmetry as well as other methods [5, 6] (an account of some interesting cases is given by Polyanin [17]). But for the first time, we generalize the equations described above to the nonlinear heat conductivity equations in the form (1.1) to investigate their symmetry properties. In order to determine more symmetry of HCEs, after finding the point symmetry group, we use preliminary group classification to find different cases of one-dimensional extension of the symmetry algebra.

The more general class of HCEs is the nonlinear heat conductivity equations of the form

$$u_t = F(t,x,u,u_x)u_{xx} + G(t,x,u,u_x), \tag{1.2}$$

which admits non-trivial symmetry group. The group classification of (1.2) is presented in some references [1, 12]. However, since the equivalence group of (1.2) is essentially wider than those for particular cases, the results of [1, 12] cannot be directly used for symmetry classification of particular ones. Nevertheless, these results are useful for finding additional equivalence transformations in the class of our problem. Therefore in contrast to the above works, in the last two sections of this paper, we study group classification of Eq. (1.1) under equivalence transformations in the general case. Furthermore, a number of nonlinear invariant models which have nontrivial invariance algebras are obtain.

From [21] we know that if the partial differential equation possesses non-trivial symmetry, then it is invariant under some finite-dimensional Lie algebra of differential operators which is completely determined by its structural constants. In the event that the maximal algebra of invariance is infinite–dimensional, then it contains, as a rule, some finite-dimensional Lie algebra. Also, if there are local non-singular changes of variables which transform a given differential equation into another, then the finite-dimensional Lie algebra of invariance of these equations are isomorphic, and in the group-theoretic analysis of differential equations such equations are considered to be equivalent. To realize the group classification, we use of the proposed approach consists in the implementation of an algorithm explained and performed in references [1, 9, 15, 18]. For this goal, our method is similar to the way of [11] for the nonlinear wave equation $u_{tt} = f(x,u)u_{xx} + g(x,u)$.

In the next section, we concern with the problem of finding point symmetry group of Eq. (1.1). In reminded sections of the paper we find some further symmetry properties of Eq. (1.1) by use of equivalence transformations and an extension by one dimension of the principal Lie algebra of the equation.

2 Lie point symmetries

In this section, our study is based on the method of [14] for Lie infinitesimal method. we are concerning with group classification of HCEs by the point transformations group.

An equation of class (1.1) is a relation among with the variables of 2–jet space $J^2(\mathbf{R}^2,\mathbf{R})$ with (local) coordinate

$$(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}), \tag{2.3}$$

where this coordinate involving independent variables t, x and dependent variable u and derivatives of u in respect to t and x up to order 2 (each index will indicate the derivation with respect to it, unless we specially state otherwise). Let $\mathcal{M}$ be the total space of independent and dependent variables resp. t, x and u. The solution space of Eq. (1.1), (if it exists) is a subvariety $S_\Delta \subset J^2(\mathbf{R}^2,\mathbf{R})$ of the second order jet bundle of 2-dimensional sub-manifolds of $\mathcal{M}$. Point symmetry group on $\mathcal{M}$ is introduced by transformations in the form of

$$\tilde{t} = \Theta(t, x, u), \qquad \tilde{x} = \Xi(t, x, u), \qquad \tilde{u} = \Omega(t, x, u), \tag{2.4}$$

for arbitrary smooth functions φ, χ, ψ. Also assume that the general form of infinitesimal generators is

$$Y := \xi(t, x, u)\,\frac{\partial}{\partial t} + \tau(t, x, u)\,\frac{\partial}{\partial x} + \varphi(t, x, u)\,\frac{\partial}{\partial u}, \tag{2.5}$$

when coefficients are arbitrary smooth functions. These infinitesimals signify the Lie algebra $\mathcal{L}$ of the point symmetry group G of Eq. (1.1). The second order prolongation of X [14, 15] as a vector field on $J^2(\mathbf{R}^2,\mathbf{R})$ is as follows

$$Y^{(2)} := v + \varphi^t\,\frac{\partial}{\partial u_t} + \varphi^x\,\frac{\partial}{\partial u_x} + \varphi^{tt}\,\frac{\partial}{\partial u_{tt}} + \varphi^{tx}\,\frac{\partial}{\partial u_{tx}} + \varphi^{xx}\,\frac{\partial}{\partial u_{xx}}, \tag{2.6}$$

where η^t, η^x and $\eta^{tt}, \eta^{tx}, \eta^{xx}$ are arbitrary smooth functions depend to variables t, x, u, u_t, u_x and (2.3) resp. These coefficients are introduced as following

$$\eta^J \;\; = \;\; \mathcal{D}_J(Q) + \xi\, u_{J,t} + \tau\, u_{J,x} \tag{2.7}$$

where $\mathcal{D}$ is total derivative, J is a multi-index with length $1 \le |J| \le 2$ of variables t, x and $Q = u - \xi\, u_t - \tau\, u_x$ is characteristic of v [14]. According to [14], v is a point infinitesimal generator of Eq. (1.1) if and only if $Y^{(2)}$[Eq. (1.1)] $= 0$ when Eq. (1.1) is hold. By applying $Y^{(2)}$ on the equation we have the following equation

$$\begin{aligned}&\tau[E_x\, u_{xx} + E_u\, u_x\, u_{xx} + E_{xx}\, u_x^2 + E_{uu}\, u_x^3 + E_{xx}\, u_x + E_{ux}\, u_x^2 + H_u\, u_x + H_x] + \varphi[E_u\, u_{xx} + E_{uu}\, u_x^2 \\ &+ E_{ux}\, u_x + H_u] - \varphi^t + \varphi^x[2E_u\, u_x + E_x] + E\,\varphi^{xx} = 0, \qquad \text{whenever} \quad \text{Eq. (1.1) is satisfied.} \qquad (2.8)\end{aligned}$$

In the extended form of the latter equation when we consider $u_t = [E(x,u)u_x]_x + H(x,u)$, functions ξ, τ and φ only depend to t, x, u rather than other variables u_t, u_x, u_{tt}, u_{tx} and u_{xx}, hence the equation will be satisfied if and only if the individual coefficients of the powers of u_t, u_x and their multiplications vanish. This tends to the following over-determined system of determining equations

$$\begin{aligned}&E\,\xi_x = 0, \qquad E\,\xi_u = 0, \qquad E_u\,\tau + 2E\,\tau_u = 0,\\ &E\,\tau_{uu} + E_u\,\tau_u - E_{uu}\,\tau = 0, \qquad E_x\,\tau + E_u\,\varphi - 2E\,\tau_x + E\,\xi_t = 0,\\ &H_x(\tau+\varphi) - \varphi_t + E_x\,\varphi_x + E\,\varphi_{xx} + H(\xi_t - \varphi_u) = 0,\\ &E_u\,\xi_t + E(\varphi_{uu} - 2\tau_{ux}) - 2E_u\,\tau_x + E_u\,\varphi_u + E_{uu}\,\varphi + 2E_{ux}\,\tau = 0,\\ &(E_{xx} + 2H_u)\tau + E_{ux}\,\varphi + \tau_t + 2E_u\,\varphi_x + E_x(\xi_t - \tau_x) + E(2\,\varphi_{ux} - \tau_{xx}). \qquad (2.9)\end{aligned}$$

The general solution to differential equations (2.9) for ξ, τ and φ is

$$\xi(t,x,u) = c, \qquad \tau(t,x,u) = 0, \qquad \varphi(t,x,u) = 0, \qquad (2.10)$$

for arbitrary constant c.

Theorem 1. *A complete set of all infinitesimal generators of the HCE (1.1) up to point transformations admits the structure of one-dimensional Lie algebra* $\mathcal{L} = \langle \frac{\partial}{\partial t} \rangle$.

It is well–known that the existence of a non–fiber–preserving symmetry usually indicates that one can significantly simplify the equation by some kind of hodograph–like transformation interchanging the independent and dependent variables. Since the kernel of maximal Lie algebra of the HCE (1.1) is $\mathcal{L} = \langle \frac{\partial}{\partial t} \rangle$, thus we can not use this advantage for simplifying HCEs.

Furthermore, according to the statements of page 209 of [14], we conclude that

Corollary 2. *A system of HCEs of class (1.1) can not be reduced into an inhomogeneous form of a linear system.*

3 Equivalence transformations

In this section, we follow the method of Ovsiannikov [15] for partial differential equations. His approach is based on the concept of an equivalence group, which is a Lie transformation group acting in the extended space of independent variables, functions and their derivatives, and preserving the class of partial differential equations under study. It is possible to modify Lie's algorithm in order to make it applicable for the computation of this group [1, 9, 15, 18]. Next we construct the optimal system of subgroups of the equivalence group.

An *equivalence transformation* is a non-degenerate change of the variables t, x, u taking any equation of the form (1.1) into an equation of the same form, generally speaking, with different $E(x,u)$ and $H(x,u)$. The set of all equivalence transformations forms an equivalence group G. We shall find a continuous subgroup G_C of it making use of the infinitesimal method.

We investigate for an operator of the group G_C in the general form

$$Y := \xi(t,x,u)\,\frac{\partial}{\partial t} + \tau(t,x,u)\,\frac{\partial}{\partial x} + \varphi(t,x,u)\,\frac{\partial}{\partial u} + \chi(t,x,u,E,h)\,\frac{\partial}{\partial E} + \eta(t,x,u,E,h)\,\frac{\partial}{\partial h}. \qquad (3.11)$$

from the invariance conditions of Eq. (1.1) written as the system

$$u_t = [E(x,u)u_x]_x + H(x,u), \qquad E_t = H_t = 0, \tag{3.12}$$

where we assumed that u, E, H are differential variables: u on the base space (t,x) and E, H on the total space (t,x,u). Also in Eq. (3.11) the coefficients are dependent to t,x,u and the two last ones, in addition, depend to E, h. The invariance conditions of the system (3.12) are

$$\widetilde{Y}\,[u_t - [E(x,u)u_x]_x - H(x,u)] = 0, \qquad \widetilde{Y}\,[E_t] = \widetilde{Y}\,[H_t] = 0, \tag{3.13}$$

where

$$\widetilde{Y} := Y + \varphi^t\,\frac{\partial}{\partial u_t} + \varphi^x\,\frac{\partial}{\partial u_x} + \varphi^{xx}\,\frac{\partial}{\partial u_{xx}} + \chi^t\,\frac{\partial}{\partial E_t} + \chi^x\,\frac{\partial}{\partial E_x} + \chi^u\,\frac{\partial}{\partial E_u} + \eta^t\,\frac{\partial}{\partial H_t}. \tag{3.14}$$

is the prolongation of the operator (3.11). Coefficients η^J for multi–index J (with length $1 \le |J| \le 2$) have given in section 2 and by applying the prolongation procedure to differential variables E, H with independent variables (t,x,u) we have

$$\begin{aligned}\chi^I &= \widetilde{\mathcal{D}}_I(\chi) - E_t\,\widetilde{\mathcal{D}}_I(\xi) - E_x\,\widetilde{\mathcal{D}}_I(\tau) - E_u\,\widetilde{\mathcal{D}}_I(\phi) = \widetilde{\mathcal{D}}_I(\chi) - E_x\,\widetilde{\mathcal{D}}_I(\tau) - E_u\,\widetilde{\mathcal{D}}_I(\phi),\\ \eta^t &= \widetilde{\mathcal{D}}_t(\eta) - H_t\,\widetilde{\mathcal{D}}_t(\xi) - H_x\,\widetilde{\mathcal{D}}_t(\eta) - H_u\,\widetilde{\mathcal{D}}_t(\phi) = \widetilde{\mathcal{D}}_t(\eta) - H_x\,\widetilde{\mathcal{D}}_t(\tau) - H_u\,\widetilde{\mathcal{D}}_t(\phi),\end{aligned} \tag{3.15}$$

where I varies on variables t,x,u and

$$\widetilde{\mathcal{D}}_I := \frac{\partial}{\partial I} + E_I\,\frac{\partial}{\partial E} + H_I\,\frac{\partial}{\partial H}.$$

Substituting (3.14) in (3.13) we tend to the following system

$$\chi\,u_{xx} + \eta - \varphi^t + [2E_u\,u_x + E_x]\,\varphi^x + E\,\varphi^{xx} + \chi^x\,u_x + \chi^u\,u_x^2 = 0, \tag{3.16}$$

$$\chi^t = 0, \qquad \eta^t = 0. \tag{3.17}$$

Replacing relations η^J (for multi–index J with length $1 \le |J| \le 2$) and χ^t, χ^x in Eqs. (3.16)–(3.17) and then introducing the relation $u_t = [E(x,u)u_x]_x + H(x,u)$ to eliminate u_t, we have three relations which are called determining equations. The two last ones are the determining equations associated with Eqs. (3.17), i.e.,

$$\chi_t - E_x\,\tau_t - E_u\,\varphi_t = 0, \qquad \eta_t - H_x\,\tau_t - H_u\,\varphi_t = 0. \tag{3.18}$$

But these relations must hold for arbitrary variables E_x, E_u, H_x, H_u of the jet space and this fact results in the following conditions

$$\tau_t = 0, \qquad \varphi_t = 0, \qquad \chi_t = 0, \qquad \eta_t = 0, \tag{3.19}$$

so, we find that

$$\xi = \xi(t,x,u), \qquad \eta = \eta(x,u), \qquad \varphi = \varphi(x,u), \qquad \chi = \chi(x,u,E,H), \qquad \eta = \eta(x,u,E,H). \tag{3.20}$$

But adding these conditions to the first determining equation, knowing that $u_t, u_x, u_{tt}, u_{tx}, u_{xx}$ are considered to be independent variables, we lead to the following system of equations

$$\begin{aligned}&\chi + E\,[\xi_t - 2\,\tau_x] = 0, \qquad E\,\varphi_{uu} + \chi_u = 0, \qquad \xi_t - 2\tau_x + \chi_E = 0,\\ &E\,[2\,\varphi_{ux} - \tau_{xx}] + \chi_x = 0, \qquad \varphi_x = 0, \qquad \chi_H = 0,\\ &\eta + H\,[\xi_t - \varphi_u] + E\,\varphi_{xx} = 0, \qquad E\,\xi_x = 0, \qquad E\,\xi_u = 0, \qquad E\,\tau_u = 0.\end{aligned} \tag{3.21}$$

This system follows

$$\begin{aligned}&\xi = (2\,c_1 + c_2)\,t + c_3, \qquad \tau = c_1\,x + c_4, \qquad \varphi = c_5\,u + c_6,\\ &\chi = c_2\,E, \qquad \eta = (c_5 - 2\,c_1 - c_2)\,H,\end{aligned} \tag{3.22}$$

for arbitrary constants $c_1, \cdots, c_6$. Therefore the class of Eqs. (1.1) has an infinite continuous group of equivalence transformations generated by infinitesimal operators

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial t}, \qquad Y_2 = \frac{\partial}{\partial x}, \qquad Y_3 = \frac{\partial}{\partial u}, \qquad Y_4 = 2\,t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial x} - 2\,H\,\frac{\partial}{\partial H}, \\ &Y_5 = -t\,\frac{\partial}{\partial t} + E\,\frac{\partial}{\partial E} + H\,\frac{\partial}{\partial H}, \qquad Y_6 = u\,\frac{\partial}{\partial u} + H\,\frac{\partial}{\partial H}. \end{aligned} \tag{3.23}$$

Moreover, in the group of equivalence transformations are included also discrete transformations, i.e., reflections

$$t \longmapsto -t, \qquad u \longmapsto -u, \qquad E \longmapsto -E, \qquad H \longmapsto -H. \tag{3.24}$$

The communication relations between these vector fields is given in Table 1. The Lie algebra $\mathcal{L} := \langle\, Y_i : i = 1, \cdots, 6\,\rangle$ is solvable since the descending sequence of derived subalgebras of $\mathcal{L}$: $\mathcal{L} \supset \mathcal{L}^{(1)} = \langle\, Y_1, Y_2, Y_3\,\rangle \supset \mathcal{L}^{(2)} = \{0\}$, terminates with a null ideal. But for each $v = \sum_i v_i Y_i$ and $w = \sum_j w_j Y_j$ in $\mathcal{L}$, its Killing form:

$$K(v, w) = \mathrm{tr}(\mathrm{ad}(v) \circ \mathrm{ad}(w)) = 5\,a_4\,b_4 - 2\,(a_4\,b_5 + a_5\,b_4) + a_5\,b_5 + a_6\,b_6, \tag{3.25}$$

is degenerate. Hence $\mathcal{L}$ is neither semisimple nor simple.

Theorem 3. *Let G_i be the one–parameter group (flow) generated by Y_i, then we have*

$$\begin{array}{ll} G_1 : (t, x, u, E, h) \longmapsto (t + s, x, u, E, H), & G_2 : (t, x, u, E, h) \longmapsto (t, x + s, u, E, H), \\ G_3 : (t, x, u, E, h) \longmapsto (t, x, u + s, E, H), & G_4 : (t, x, u, E, h) \longmapsto (t\,e^{2s}, x\,e^{s}, u, E, H\,e^{-2s}), \\ G_5 : (t, x, u, E, h) \longmapsto (t\,e^{-s}, x, u, E\,e^{s}, H\,e^{s}), & G_6 : (t, x, u, E, h) \longmapsto (t, x, u\,e^{s}, E, H\,e^{s}), \end{array} \tag{3.26}$$

when s is an arbitrary parameter. Moreover, if $u = f(t, x)$ for functions E and H be a solution of the HCE (1.1), so are

$$u_1 = f(t + s, x), \qquad u_2 = f(t, x + s), \qquad u_3 = f(t, x, u) - s \tag{3.27}$$

for the same functions E and H, $u_4 = f(t\,e^{2s}, x\,e^{s})$ for the same functions E and $\bar{H} = H\,e^{-2s}$, $u_5 = f(t\,e^{-s}, x)$ for $\bar{E} = E\,e^{s}$ and $\bar{H} = H\,e^{s}$ and also $u_6 = e^{-s}\,f(t, x)$ for the same E and $\bar{H} = H\,e^{s}$.

4 Preliminary group classification

In many applications of group analysis, most of extensions of the principal Lie algebra admitted by the equation under consideration are taken from the equivalence algebra $\mathcal{L}_{\mathcal{E}}$. These extensions are called $\mathcal{E}$–extensions of the principal Lie algebra. The classification of all nonequivalent equations (with respect to a given equivalence group $G_{\mathcal{E}}$) admitting $\mathcal{E}$–extensions of the principal Lie algebra is called a *preliminary group classification* [9]. We consider the algebra $\mathcal{L}$ spanned on operators (3.23) and use it for a preliminary group classification.

It is well-known that the problem of classifying invariant solutions is equivalent to the problem of classifying subgroups of the full symmetry group under conjugation in which itself is equivalent to determining all conjugate subalgebras [14, 15]. The latter problem, tends to determine a list (that is called an *optimal system*) of conjugacy inequivalent subalgebras with the property that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation i.e. $\overline{\mathcal{L}}_H\,\mathrm{Ad}(g)\,\mathcal{L}_H$ for some g of a considered Lie group. Thus we will deal with the construction of the optimal system of subalgebras of $\mathcal{L}$.

The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(s\,Y_i))Y_j = Y_j - s\,[Y_i, Y_j] + \frac{s^2}{2}\,[Y_i, [Y_i, Y_j]] - \cdots, \tag{4.28}$$

where s is a parameter and $i, j = 1, \cdots, 6$. The adjoint representations of $\mathcal{L}$ is listed in Tables 2; it consists the separate adjoint actions of each element of $\mathcal{L}$ on all other elements.

Table 1: Commutator table

[,]	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	0	0	0	$2\,Y_1$	$-Y_1$	0
Y_2	0	0	0	Y_2	0	0
Y_3	0	0	0	0	0	Y_3
Y_4	$-2\,Y_1$	$-Y_2$	0	0	0	0
Y_5	Y_1	0	0	0	0	0
Y_6	0	0	$-Y_3$	0	0	0

Table 2: Adjoint table

Ad	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	Y_1	Y_2	Y_3	$Y_4 - 2s\,Y_1$	$Y_5 + s\,Y_1$	Y_6
Y_2	Y_1	Y_2	Y_3	$Y_4 - s\,Y_2$	Y_5	Y_6
Y_3	Y_1	Y_2	Y_3	Y_4	Y_5	$Y_6 - s\,Y_3$
Y_4	$e^{2s}\,Y_1$	$e^{s}\,Y_2$	Y_3	Y_4	Y_5	Y_6
Y_5	$e^{-s}\,Y_1$	Y_2	Y_3	Y_4	Y_5	Y_6
Y_6	Y_1	Y_2	$e^{s}\,Y_3$	Y_4	Y_5	Y_6

Theorem 4. *An optimal system of one-dimensional Lie subalgebras of the nonlinear HCE (1.1) is provided by those generated by*

$$
\begin{aligned}
&1)\ A^1 = Y_1 = \partial_t,\\
&2)\ A^2 = Y_2 = \partial_x,\\
&3)\ A^3 = Y_3 = \partial_u,\\
&4)\ A^4 = Y_4 = 2t\,\partial_t + \partial_x - 2H\,\partial_H,\\
&5)\ A^5 = Y_5 = -t\,\partial_t + E\,\partial_E + H\,\partial_H,\\
&6)\ A^6 = Y_6 = u\,\partial_u + H\,\partial_H,\\
&7)\ A^7 = \pm Y_1 + Y_2 = \pm\partial_t + \partial_x,\\
&8)\ A^8 = \pm Y_1 + Y_3 = \pm\partial_t + \partial_u,\\
&9)\ A^9 = \pm Y_1 + Y_6 = \pm\partial_t + u\,\partial_u + H\,\partial_H,\\
&10)\ A^{10} = Y_2 + Y_3 = \partial_x + \partial_u,\\
&11)\ A^{11} = \pm Y_2 + Y_4 = 2t\,\partial_t \pm\ \partial_x + \partial_x - 2H\,\partial_H,\\
&12)\ A^{12} = \pm Y_2 + Y_5 = -t\,\partial_t \pm \partial_x + E\,\partial_E + H\,\partial_H,\\
&13)\ A^{13} = Y_2 + Y_6 = \partial_x + u\,\partial_u + H\,\partial_H,\\
&14)\ A^{14} = \pm Y_3 + Y_4 = 2t\,\partial_t + \partial_x \pm \partial_u - 2H\,\partial_H,\\
&15)\ A^{15} = \pm Y_3 + Y_5 = -t\,\partial_t \pm \partial_u + E\,\partial_E + H\,\partial_H,\\
&16)\ A^{16} = \alpha_1\,Y_4 + Y_5 = (2\alpha_1 - 1)t\,\partial_t + \alpha_1\,\partial_x + E\,\partial_E - (2\alpha_1 - 1)H\,\partial_H,\\
&17)\ A^{17} = \alpha_2\,Y_4 + Y_6 = 2\,\alpha_2 t\,\partial_t + \alpha_2\,\partial_x + u\,\partial_u - (2\alpha_2 - 1)H\,\partial_H,\\
&18)\ A^{18} = \beta_1\,Y_5 + Y_6 = -\beta_1 t\,\partial_t + u\,\partial_u + \beta_1 E\,\partial_E + (\beta_1 + 1)H\,\partial_H,\\
&19)\ A^{19} = \pm Y_1 + Y_2 + Y_3 = \pm\partial_t + \partial_x + \partial_u,\\
&20)\ A^{20} = \pm Y_1 + Y_2 + Y_6 = \pm\partial_t + \partial_x + u\,\partial_u + H\,\partial_H,\\
&21)\ A^{21} = \pm Y_2 \pm Y_3 + Y_4 = 2t\,\partial_t \pm\ \partial_x + \partial_x \pm \partial_u - 2H\,\partial_H,\\
&22)\ A^{22} = \pm Y_2 \pm Y_3 + Y_5 = -t\,\partial_t \pm \partial_x \pm \partial_u + E\,\partial_E + H\,\partial_H,
\end{aligned}
\tag{4.29}
$$

23) $A^{23} = \pm Y_2 + \alpha_3\, Y_4 + Y_5 = (2\alpha_3 - 1)t\,\partial_t + (\alpha_3 \pm 1)\partial_x + E\,\partial_E - (2\alpha_3 - 1)H\,\partial_H,$
24) $A^{24} = Y_2 + \alpha_4\, Y_4 + Y_6 = 2\alpha_4 t\,\partial_t + (\alpha_4 + 1)\partial_x + u\,\partial_u - (2\alpha_4 - 1)H\,\partial_H,$
25) $A^{25} = \pm Y_2 + \beta_2\, Y_5 + Y_6 = -\beta_2 t\,\partial_t \pm \partial_x + u\,\partial_u + \beta_2\, E\partial_E + (\beta_2 + 1)H\,\partial_H,$
26) $A^{26} = \pm Y_3 + \alpha_5\, Y_4 + Y_5 = (2\alpha_5 - 1)t\,\partial_t + \alpha_5\,\partial_x \pm \partial_u + E\,\partial_E - (2\alpha_5 - 1)H\,\partial_H,$
27) $A^{27} = \alpha_6\, Y_4 + \beta_3\, Y_5 + Y_6 = (2\alpha_6 - \beta_3)t\,\partial_t + \alpha_6\,\partial_x + u\,\partial_u + \beta_3 E\,\partial_E - (2\alpha_6 - \beta_3 - 1)H\,\partial_H,$
28) $A^{28} = \pm Y_2 \pm Y_3 + \alpha_7\, Y_4 + Y_5 = (2\alpha_7 - 1)t\,\partial_t + (\alpha_7 \pm 1)\,\partial_x \pm \partial_u + E\,\partial_E - (2\alpha_7 - 1)H\,\partial_H,$
29) $A^{29} = \pm Y_2 + \alpha_8\, Y_4 + \beta_4\, Y_5 + Y_6 = (2\alpha_8 - \beta_4)t\,\partial_t + (\alpha_8 \pm 1)\,\partial_x + u\,\partial_u + \beta_4 E\,\partial_E - (2\alpha_8 - \beta_4 - 1)H\,\partial_H,$

for nonzero constants α_i, β_j *(*$1 \le i \le 8, 1 \le j \le 4$*).*

Proof. Let $\mathcal{L}$ is the symmetry algebra of Eq. (1.1) with adjoint representation determined in Table 2 and

$$Y = a_1\, Y_1 + a_2\, Y_2 + a_3\, Y_3 + a_4\, Y_4 + a_5\, Y_5 + a_6\, Y_6 \tag{4.30}$$

is a nonzero vector field of $\mathcal{L}$. We will simplify as many of the coefficients a_i as possible through proper adjoint applications on Y. We follow our aim in the below easy cases.

Case 1 At first, assume that $a_6 \neq 0$. Scaling Y if necessary, we can consider a_6 to be 1 and so follow the problem with

$$Y = a_1\, Y_1 + a_2\, Y_2 + a_3\, Y_3 + a_4\, Y_4 + a_5\, Y_5 + Y_6. \tag{4.31}$$

Case 1a According to Table 2 in the case which $a_4 \neq 0$, if we act on Y by $\mathrm{Ad}(\exp(\frac{a_1}{a_4}\, Y_2))$, the coefficient of Y_1 can be vanished:

$$Y' = a_2\, Y_2 + a_3\, Y_3 + a_4\, Y_4 + a_5\, Y_5 + Y_6. \tag{4.32}$$

Then for $a_3 \neq 0$ we apply $\mathrm{Ad}(\exp(a_3\, Y_3))$ on Y' to cancel the coefficient of Y_3 (it is automatically hold for $a_3 = 0$):

$$Y'' = a_2\, Y_2 + a_4\, Y_4 + a_5\, Y_5 + Y_6. \tag{4.33}$$

Case 1a-1 If in addition $a_2 \neq 0$, we can act $\mathrm{Ad}(\exp(\pm \ln(\frac{1}{a_2})\, Y_4))$ on Y'' to change a_2 to ± 1:

$$Y''' = \pm Y_2 + a_4\, Y_4 + a_5\, Y_5 + Y_6. \tag{4.34}$$

Then for the case which $a_5 \neq 0$, we can not simplify Y''' any more. This introduce part 29 of the theorem.
Also when $a_5 = 0$ we tend to part 24 of the theorem.

Case 1a-2 In Case 1a, let we consider a_4 of Y'' is equal to zero. Then for

$$Y'''' = a_4\, Y_4 + a_5\, Y_5 + Y_6, \tag{4.35}$$

where $a_5 \neq 0$ more simplification is impossible and this results in part 27, and where $a_5 = 0$ it results in part 17 of the theorem.

Case 1b Let the coefficient a_4 in Y is zero. So we have the following new form of Y

$$\overline{Y} = a_1\, Y_1 + a_2\, Y_2 + a_3\, Y_3 + a_5\, Y_5 + Y_6. \tag{4.36}$$

Case 1b-1 In the case which $a_5 \neq 0$, by applying $\mathrm{Ad}(\exp(-\frac{a_1}{a_5})\, Y_1))$ on $\overline{Y}$, the coefficient a_1 will be vanished:

$$\overline{\overline{Y}} = a_2\, Y_2 + a_3\, Y_3 + a_5\, Y_5 + Y_6. \tag{4.37}$$

Furthermore, if we act $\mathrm{Ad}(\exp(\frac{a_3}{a_6})\, Y_3))$ on the latter form, so the coefficient a_3 will be zero. When $a_2 \neq 0$, applying $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_2})\, Y_4))$ any one–dimensional subalgebra generated by Y is equivalent to one generated by $\pm Y_2 + a_5\, Y_5 + Y_6$ which introduce part 25 of the theorem for constant $a_5 \neq 0$. For the other case which $a_2 = 0$, part 18 are given.

Case 1b-2 In $\overline{Y}$, let $a_5 = 0$. In this case, by applying $\mathrm{Ad}(\exp(\frac{a_3}{a_6})\, Y_6))$ we can make the coefficient of Y_3 equal to zero:

$$\overline{\overline{\overline{Y}}} = a_1\, Y_1 + a_2\, Y_2 + Y_6. \tag{4.38}$$

Case 1b-2-1 For $a_2 \neq 0$ the action of $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_2})\, Y_4))$ on $\overline{\overline{\overline{Y}}}$ results in the form $\pm\frac{2a_1}{a_2}\, Y_1 + Y_2 + Y_6$. Then if $a_1 \neq 0$, we can apply $\mathrm{Ad}(\exp(\pm \ln \frac{a_2}{2a_1})\, Y_5))$ on it to have the form $\pm Y_1 + Y_2 + Y_6$ that is part 20 of the theorem. If we consider the case in which $a_1 = 0$, then we tend to part 13.

Case 1b-2-2 In $\overline{\overline{\overline{Y}}}$ assume that $a_2 = 0$. In this case the condition $a_1 \neq 0$ and the action of $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_1})\, Y_5))$ show that the simplest form of Y is similar to part 9, while the condition $a_1 = 0$ leads to part 6 of the theorem.

Case 2 The remaining one–dimensional subalgebras are spanned by vector fields of the form Y with $a_6 = 0$.

Case 2a If $a_5 \neq 0$ then by scaling Y, we can assume that $a_5 = 1$:

$$\hat{Y} = a_1\, Y_1 + a_2\, Y_2 + a_3\, Y_3 + a_4\, Y_4 + Y_5. \tag{4.39}$$

Case 2a-1 Now by the action of $\mathrm{Ad}(\exp(\frac{a_1}{a_4}\, Y_2))$ on Y where $a_4 \neq 0$, we can cancel the coefficient of Y_1:

$$\hat{Y}' = a_2\, Y_2 + a_3\, Y_3 + a_4\, Y_4 + Y_5. \tag{4.40}$$

Case 2a-1-1 Then for $a_3 \neq 0$ by applying $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_3}\, Y_6))$ on $\hat{Y}'$ the coefficient of Y_3 can be ± 1. Thus if $a_2 \neq 0$ the action of $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_2}\, Y_4))$ recommends part 28 of the theorem. Moreover, where $a_2 = 0$ we tend to part 26.

Case 2a-1-2 In case 2a-1, let $a_3 = 0$. So for $a_2 \neq 0$ applying $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_2}\, Y_4))$ on $\bar{Y}'$ one can lead to part 23 of the theorem, whereas for $a_2 = 0$, part 16 are achieved.

Case 2a-2 Suppose that a_4 in $\hat{Y}$ is equal to zero, then for $a_1 \neq 0$ by acting $\mathrm{Ad}(\exp(-\frac{a_1}{a_5}\, Y_5))$ and also for $a_1 = 0$ we lead to the following form

$$\hat{Y}'' = a_2\, Y_2 + a_3\, Y_3 + Y_5. \tag{4.41}$$

Case 2a-2-1 For $a_3 \neq 0$ we can act $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_6}\, Y_6))$ on $\hat{Y}''$ to change the coefficient of Y_3 to be equal to either $+1$ or -1. Hence for $a_2 \neq 0$ by applying $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_2}\, Y_4))$ and for $a_2 = 0$ we find part 22 and part 15 resp.

Case 2a-2-2 Assume that in $\hat{Y}$, a_3 is zero, then for $a_2 \neq 0$ by applying $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_2}\, Y_4))$ and for $a_2 = 0$ we tend to part 12 and part 5 of the theorem resp.

Case 2b In the situation of Case 2, suppose that $a_5 = 0$.

Case 2b-1 If in addition we assume that $a_4 \neq 0$, then if necessary we can let it equal to 1:

$$\widetilde{Y} = a_1\, Y_1 + a_2\, Y_2 + a_3\, Y_3 + Y_4. \tag{4.42}$$

So for $a_1 \neq 0$ by applying $\mathrm{Ad}(\exp(\frac{a_1}{a_4}\, Y_2))$ and also for $a_1 = 0$, we have the following form for $\widetilde{Y}$

$$\widetilde{Y}' = a_2\, Y_2 + a_3\, Y_3 + Y_4. \tag{4.43}$$

Case 2b-1-1 If $a_3 \neq 0$, the action of $\mathrm{Ad}(\pm \ln \exp(\frac{1}{a_3} Y_6))$ shows that we can make the coefficient of Y_3 equal to ± 1. Then for $a_2 \neq 0$ by applying $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_2} Y_4))$ and for $a_2 = 0$ resp. part 21 and part 14 of the theorem are given.

Case 2b-1-2 In $\widetilde{Y}'$, suppose that $a_3 = 0$. The simplest possible form of Y is equal to part 11 after taking $a_2 \neq 0$ and after acting $\mathrm{Ad}(\pm \ln \exp(\frac{1}{a_2} Y_4))$. Moreover, when $a_2 = 0$ the simplest case is equal to part 4.

Case 2b-2 With conditions of Case 2b, in addition let $a_4 = 0$:

$$\dot{Y} = a_1 Y_1 + a_2 Y_2 + a_3 Y_3. \tag{4.44}$$

Case 2b-2-1 Consider $a_3 \neq 0$, then by scaling we can make the coefficient of $\widehat{Y}$ equal to 1. By assuming $a_2 \neq 0$ and applying $\mathrm{Ad}(\exp(\pm \ln \frac{1}{a_2} Y_4))$ we find the following form

$$\dot{Y}' = \pm \frac{2a_1}{a_2} Y_1 + Y_2 + Y_3. \tag{4.45}$$

When $a_1 \neq 0$ we can reduce it to part 19 after acting $\mathrm{Ad}(-\exp(\frac{a_2}{2a_1} Y_5))$, whereas for $a_1 = 0$ we find part 10. If we change our assumption on a_2 and consider it equal to 0, for $a_1 \neq 0$ by applying $\mathrm{Ad}(\pm \ln \exp(\frac{1}{a_1} Y_5))$ and for $a_1 = 0$ resp. we lead to part 8 and part 3.

Case 2b-2-2 Finally if in the same conditions with Case 2b-2, we assume that a_3 be zero, by scaling we can make the coefficient of Y_2 equal to 1 if $a_2 \neq 0$:

$$\dot{Y}'' = a_1 Y_1 + Y_2. \tag{4.46}$$

If $a_1 \neq 0$ we can act $\mathrm{Ad}(\pm \ln \exp(-\ln \frac{1}{a_1} Y_5))$ on $\dot{Y}''$. Then no further simplification is possible and then Y is reduced to part 7, while $a_1 = 0$ suggests part 2. The last case occurs when we change the condition on a_1 to be equal to zero, which recommends part 1 of the theorem.

There is not any more possible case for studying and the proof is complete (Note that in the group of equivalence transformations there are included also discrete transformations, Eq. (3.24)). $\boldsymbol{\Lambda}$

The coefficients E, H of Eq. (1.1) depend on the variables x, u. Therefore, we take their optimal system's projections on the space (x, u, E, h). The nonzero in $(x, u)-$axis projections of (4.29) are

$$\begin{aligned}
&1)\ Z^1 = A^2 = A^7 = \partial_x,\\
&2)\ Z^2 = A^3 = A^8 = \partial_u,\\
&3)\ Z^3 = A^4 = \partial_x - 2H\,\partial_H,\\
&4)\ Z^4 = A^5 = E\,\partial_E + H\,\partial_H,\\
&5)\ Z^5 = A^6 = A^9 = u\,\partial_u + H\,\partial_H,\\
&6)\ Z^6 = A^{10} = A^{19} = \partial_x + \partial_u,\\
&7)\ Z^7 = A^{11} = \pm\partial_x + \partial_x - 2H\,\partial_H,\\
&8)\ Z^8 = A^{12} = \pm\partial_x + E\,\partial_E + H\,\partial_H,\\
&9)\ Z^9 = A^{13} = A^{20} = \partial_x + u\,\partial_u + H\,\partial_H,\\
&10)\ Z^{10} = A^{14} = \partial_x \pm \partial_u - 2H\,\partial_H,\\
&11)\ Z^{11} = A^{15} = \pm\partial_u + E\,\partial_E + H\,\partial_H,\\
&12)\ Z^{12} = A^{16} = \alpha_1\,\partial_x + E\,\partial_E - (2\alpha_1 - 1)H\,\partial_H,\\
&13)\ Z^{13} = A^{17} = \alpha_2\,\partial_x + u\,\partial_u - (2\alpha_2 - 1)H\,\partial_H,\\
&14)\ Z^{14} = A^{18} = u\,\partial_u + \beta_1 E\,\partial_E + (\beta_1 + 1)H\,\partial_H,\\
&15)\ Z^{15} = A^{21} = \pm\partial_x + \partial_x \pm \partial_u - 2H\,\partial_H,\\
&16)\ Z^{16} = A^{22} = \pm\partial_x \pm \partial_u + E\,\partial_E + H\,\partial_H,
\end{aligned} \tag{4.47}$$

17) $Z^{17} = A^{23} = (\alpha_3 \pm 1)\partial_x + E\,\partial_E - (2\alpha_3 - 1)H\,\partial_H,$
18) $Z^{18} = A^{24} = (\alpha_4 + 1)\partial_x + u\,\partial_u - (2\alpha_4 - 1)H\,\partial_H,$
19) $Z^{19} = A^{25} = \pm\partial_x + u\,\partial_u + \beta_2\,E\partial_E + (\beta_2 + 1)H\,\partial_H,$
20) $Z^{20} = A^{26} = \alpha_5\,\partial_x \pm \partial_u + E\,\partial_E - (2\alpha_5 - 1)H\,\partial_H,$
21) $Z^{21} = A^{27} = \alpha_6\,\partial_x + u\,\partial_u + \beta_3 E\,\partial_E - (2\alpha_6 - \beta_3 - 1)H\,\partial_H,$
22) $Z^{22} = A^{28} = (\alpha_7 \pm 1)\,\partial_x \pm \partial_u + E\,\partial_E - (2\alpha_7 - 1)H\,\partial_H,$
23) $Z^{23} = A^{29} = (\alpha_8 \pm 1)\,\partial_x + u\,\partial_u + \beta_4 E\,\partial_E - (2\alpha_8 - \beta_4 - 1)H\,\partial_H,$

From paper 7 of [9] we conclude that

Proposition 5. *Let $\mathcal{L}_m := \langle Y_i \,:\, i = 1, \cdots, m\rangle$ be an m–dimensional algebra. Denote by $A^i\,(i = 1, \cdots, s,\ 0 < s \leq m,\ s \in \mathbb{N})$ an optimal system of one–dimensional subalgebras of $\mathcal{L}_m$ and by $Z^i\,(i = 1, \cdots, t,\ 0 < t \leq s,\ t \in \mathbb{N})$ the projections of A^i, i.e., $Z^i = \mathrm{pr}(A^i)$. If equations*

$$F = F(x, u), \qquad G = G(x, u), \tag{4.48}$$

are invariant with respect to the optimal system Z^i then the equation

$$u_t = (F(x, u)\,u_x)_x + G(x, u)\,u, \tag{4.49}$$

admits the operators X^i = projection of A^i on (t, x, u).

Proposition 6. *Let Eq. (4.49) and the equation*

$$u_t = (\overline{F}(x, u)\,u_x)_x + \overline{G}(x, u), \tag{4.50}$$

be constructed according to Proposition 4 via optimal systems Z^i and $\overline{Z}^i$ resp. If the subalgebras spanned on the optimal systems Z^i and $\overline{Z}^i$ resp. are similar in $\mathcal{L}_m$, then Eqs. (4.49) and (4.50) are equivalent with respect to the equivalence group G_m generated by $\mathcal{L}_m$.

Now by applying Propositions 4 and 4 for the optimal system (4.47), we want to find all nonequivalent equations in the form of Eq. (1.1) admitting $\mathcal{E}$–extensions of the principal Lie algebra $\mathcal{L}_{\mathcal{E}}$, by one dimension, i.e, equations of the form (1.1) such that they admit, together with the one basic operator $\frac{\partial}{\partial t}$ of $\mathcal{L}_1$, also a second operator $X^{(2)}$. In each case which this extension occurs, we indicate the corresponding coefficients E, H and the additional operator $X^{(2)}$.

We perform the algorithm passing from operators $Z^i\,(i = 1, \cdots, 23)$ to E, H and $X^{(2)}$ via the following examples.

Let consider the vector field

$$Z^4 = E\,\partial_E + H\,\partial_H, \tag{4.51}$$

then the characteristic equation corresponding to Z^4 is

$$\frac{dE}{E} = \frac{dH}{H}, \tag{4.52}$$

which determines invariants. Invariants can be taken in the following form

$$I_1 = x, \qquad I_2 = u, \qquad I_3 = \frac{H}{E}. \tag{4.53}$$

In this case there are no invariant equations because the necessary condition for existence of invariant solutions (see [15], Section 19.3) is not satisfied, i.e., invariants (4.53) cannot be solved with respect to E and H since each two of them can not be an invariant function with respect to the third one.

Table 3: The result of the classification

N	Z	Invariant λ	Equation	Additional operator $X^{(2)}$
1	Z^1	u	$u_t = [\Phi\, u_x]_x + \Psi$	$\frac{\partial}{\partial x},\ \pm\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$
2	Z^2	x	$u_t = [\Phi\, u_x]_x + \Psi$	$\frac{\partial}{\partial u},\ \pm\frac{\partial}{\partial t} + \frac{\partial}{\partial u}$
3	Z^3	u	$u_t = [\Phi\, u_x]_x + e^{-2(x+\Psi)}$	$2t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$
4	Z^5	x	$u_t = [\Phi\, u_x]_x + e^{\ln u+\Psi}$	$t\,\frac{\partial}{\partial t}$
5	Z^6	$u-x$	$u_t = [\Phi\, u_x]_x + \Psi$	$\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\ \pm\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial u}$
6	Z^7	u	$u_t = [\Phi\, u_x]_x + e^{-(x+\Psi)}$	$t\,\frac{\partial}{\partial t},\ t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$
7	Z^8	u	$u_t = [e^{\Phi\pm x}\, u_x]_x + e^{\Psi\pm x}$	$\pm t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$
8	Z^9	$\ln u - x$	$u_t = [\Phi\, u_x]_x + e^{x+\Psi}$	$\frac{\partial}{\partial x} + u\,\frac{\partial}{\partial u},\ \pm\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + u\,\frac{\partial}{\partial u}$
9	Z^{10}	$x \pm u$	$u_t = [\Phi\, u_x]_x + e^{-2(\Psi\pm u)}$	$2t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \pm \frac{\partial}{\partial u}$
10	Z^{11}	x	$u_t = [e^{\Phi\pm u}\, u_x]_x + e^{\Psi\pm u}$	$t\,\frac{\partial}{\partial t} \pm \frac{\partial}{\partial u}$
11	$Z^{12}(\alpha_1 \neq 1/2)$	u	$u_t = [e^{\Phi+\frac{1}{\alpha_1}x}\, u_x]_x + e^{\Psi+(2-\frac{1}{\alpha_1}x)}$	$(2\alpha_1-1)t\,\frac{\partial}{\partial t} + \alpha_1\,\frac{\partial}{\partial x}$
12	$Z^{12}(\alpha_1 = 1/2)$	u	$u_t = [e^{\Phi+\frac{1}{\alpha_1}x}\, u_x]_x + \Psi$	$\frac{\partial}{\partial x}$
13	$Z^{13}(\alpha_2 \neq 1/2)$	$\ln u - \frac{x}{\alpha_2}$	$u_t = [\Phi\, u_x]_x + e^{(1-2\alpha_2)(\ln u+\Psi)}$	$2\alpha_2\, t\,\frac{\partial}{\partial t} + \alpha_2\,\frac{\partial}{\partial x} + u\,\frac{\partial}{\partial u}$
14	$Z^{13}(\alpha_2 = 1/2)$	$\ln u - 2x$	$u_t = [\Phi\, u_x]_x + \Psi$	$2t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + 2u\,\frac{\partial}{\partial u}$
15	$Z^{14}(\beta_1 \neq -1)$	x	$u_t = [e^{\beta_1(\Phi(u)+\ln u)}\, u_x]_x + e^{(1+\beta_1)(\Psi+\ln u)}$	$\beta_1 t\,\frac{\partial}{\partial t} - u\,\frac{\partial}{\partial u}$
16	$Z^{14}(\beta_1 = -1)$	x	$u_t = [u\Phi\, u_x]_x + \Psi$	$t\,\frac{\partial}{\partial t} - u\,\frac{\partial}{\partial u}$
17	$Z^{15}(2\partial_x \pm \partial_u - 2H\partial_H)$	$\frac{1}{2}\,x \pm u$	$u_t = [\Phi(u)\, u_x]_x + e^{-2(\Psi\pm u)}$	$2\,t\,\frac{\partial}{\partial t} + 2\,\frac{\partial}{\partial x} \pm \frac{\partial}{\partial u}$
18	$Z^{15}(\pm\partial_u - 2H\partial_H)$	x	$u_t = [\Phi\, u_x]_x + e^{-2(\Psi\pm u)}$	$2\,t\,\frac{\partial}{\partial t} \pm \frac{\partial}{\partial u}$
19	Z^{16}	$\pm x \pm u$	$u_t = [e^{\Phi\pm u}\, u_x]_x + e^{\Psi\pm u}$	$-t\,\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x} \pm \frac{\partial}{\partial u}$
20	$Z^{17}(\alpha_3 \neq \pm 1, 1/2)$	u	$u_t = [e^{\Phi+(\alpha_3\pm 1)x}\, u_x]_x + e^{\Psi-\frac{2\alpha_3-1}{\alpha_3\pm 1}x}$	$(2\alpha_3-1)t\,\frac{\partial}{\partial t} + (\alpha_3\pm 1)\,\frac{\partial}{\partial x}$
21	$Z^{17}(\alpha_3 = 1/2)$	u	$u_t = [e^{\Phi+(1/2\pm 1)x}\, u_x]_x + \Psi$	$\frac{\partial}{\partial x}$
22	$Z^{18}(\alpha_4 \neq -1, 1/2)$	$\ln u - \frac{x}{\alpha_4+1}$	$u_t = [\Phi\, u_x]_x + e^{\Psi-\frac{2\alpha_4-1}{\alpha_4+1}x}$	$2\alpha_4\, t\,\frac{\partial}{\partial t} + (\alpha_4+1)\,\frac{\partial}{\partial x} + u\,\frac{\partial}{\partial u}$
23	$Z^{18}(\alpha_4 = -1)$	x	$u_t = [\Phi\, u_x]_x + e^{3(\Psi+\ln u)}$	$2\,t\,\frac{\partial}{\partial t} - u\,\frac{\partial}{\partial u}$
24	$Z^{18}(\alpha_4 = 1/2)$	$\ln u - \frac{2}{3}\,x$	$u_t = [\Phi\, u_x]_x + \Psi$	$t\,\frac{\partial}{\partial t} + \frac{3}{2}\,\frac{\partial}{\partial x} + u\,\frac{\partial}{\partial u}$
25	$Z^{19}(\beta_2 \neq -1)$	$\ln u \pm x$	$u_t = [e^{\beta_2(\Phi\pm x)}\, u_x]_x + e^{(\beta_2+1)(\Psi\pm x)}$	$-\beta_2 t\,\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x} + u\,\frac{\partial}{\partial u}$
26	$Z^{19}(\beta_2 = -1)$	$\ln u \pm x$	$u_t = [e^{\Phi\pm x}\, u_x]_x + \Psi$	$t\,\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x} + u\,\frac{\partial}{\partial u}$
27	$Z^{20}(\alpha_5 \neq 1/2)$	$\frac{x}{\alpha_5} \pm u$	$u_t = [e^{\Phi\pm u}\, u_x]_x + e^{\Psi-(2-\frac{1}{\alpha_5})x}$	$(2\alpha_5-1)t\,\frac{\partial}{\partial t} + \alpha_5\,\frac{\partial}{\partial x} \pm \frac{\partial}{\partial u}$
28	$Z^{20}(\alpha_5 = 1/2)$	$\frac{x}{2} \pm u$	$u_t = [e^{\Phi\pm u}\, u_x]_x + \Psi$	$\frac{\partial}{\partial x} \pm 2\,\frac{\partial}{\partial u}$
29	$Z^{21}(2\alpha_6 - \beta_3 \neq 1)$	$\ln u - \frac{x}{\alpha_6}$	$u_t = [e^{\beta_3(\Phi+\alpha_6 x)}\, u_x]_x + e^{\Psi-\frac{2\alpha_6-\beta_3-1}{\alpha_6}x}$	$(2\alpha_6-\beta_3)t\,\frac{\partial}{\partial t} + \alpha_6\,\frac{\partial}{\partial x} + u\,\frac{\partial}{\partial u}$
30	$Z^{21}(2\alpha_6 - \beta_3 = 1)$	$\ln u - \frac{x}{\alpha_6}$	$u_t = [e^{\beta_3(\Phi+\alpha_6 x)}\, u_x]_x + \Psi$	$t\,\frac{\partial}{\partial t} + \alpha_6\,\frac{\partial}{\partial x} + u\,\frac{\partial}{\partial u}$
31	$Z^{22}(\alpha_7 \neq \pm 1, 1/2)$	$\frac{x}{\alpha_7\pm 1} \pm u$	$u_t = [e^{\Phi\pm u} u_x]_x + e^{(1-2\alpha_7)(\Psi\pm u)}$	$(2\alpha_7-1)t\,\frac{\partial}{\partial t} + (\alpha_7\pm 1)\,\frac{\partial}{\partial x} \pm \frac{\partial}{\partial u}$
32	$Z^{22}(\alpha_7 = \pm 1)$	x	$u_t = [e^{\Phi\pm u}\, u_x]_x + e^{(1\pm 2)(\Psi\pm u)}$	$(\pm 2 - 1)\,t\,\frac{\partial}{\partial t} \pm \frac{\partial}{\partial u}$
33	$Z^{22}(\alpha_7 = 1/2)$	$\frac{x}{1/2\pm 1} \pm u$	$u_t = [e^{\Phi\pm u}\, u_x]_x + \Psi$	$(\frac{1}{2} \pm 1)\,\frac{\partial}{\partial x} \pm \frac{\partial}{\partial u}$
34	$Z^{23}(\alpha_8 \neq \pm 1, 2\alpha_8 - \beta_4 \neq 1)$	$\ln u - \frac{x}{\alpha_8\pm 1}$	$u_t = [e^{\beta_4(\Phi+\ln u)} u_x]_x + e^{\Psi-\frac{2\alpha_8-\beta_4-1}{\alpha_8\pm 1}x}$	$(2\alpha_8-\beta_4)t\frac{\partial}{\partial t} + (\alpha_8\pm 1)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$
35	$Z^{23}(\alpha_8 \neq \pm 1, 2\alpha_8 - \beta_4 = 1)$	$\ln u - \frac{x}{\alpha_8\pm 1}$	$u_t = [e^{\beta_4(\Phi+\ln u)}\, u_x]_x + \Psi$	$t\,\frac{\partial}{\partial t} + (\alpha_8 \pm 1)\,\frac{\partial}{\partial x} + u\,\frac{\partial}{\partial u}$
36	$Z^{23}(\alpha_8 = \pm 1, 2\alpha_8 - \beta_4 \neq 1)$	x	$u_t = [e^{\beta_4(\Phi+\ln u)}\, u_x]_x + e^{(2\alpha_8-\beta_4-1)(\Psi+\ln u)}$	$(\pm 2 - \beta_4)\,t\,\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial u}$
37	$Z^{23}(\alpha_8 = \pm 1, 2\alpha_8 - \beta_4 = 1)$	x	$u_t = [e^{(\pm 2-1)(\Phi+\ln u)}\, u_x]_x + \Psi$	$u\,\frac{\partial}{\partial u}$

Considering Z^{23} (for $a_8 \pm 1 \neq 0$, $2\alpha_8 - \beta_4 - 1 \neq 0$) we have the below characteristic equation

$$\frac{dx}{\alpha_8 \pm 1} = \frac{du}{u} = \frac{dE}{\beta_4 E} = \frac{dH}{(2\alpha_8 - \beta_4 - 1)H}, \tag{4.54}$$

This equation suggest the following invariants

$$I_1 = \ln u - (\alpha_8 \pm 1)x, \qquad I_2 = \ln E^{1/\beta_4} - \ln u, \qquad I_3 = \ln H + \frac{2\alpha_8 - \beta_4 - 1}{\alpha_8 \pm 1}\, x. \tag{4.55}$$

From the invariance equations we can write

$$I_2 = \Phi(I_1), \qquad I_3 = \Psi(I_1), \tag{4.56}$$

They result in the forms

$$E = \exp(\beta_4(\ln u + \Phi(\lambda))), \qquad H = \exp\Big(\Psi(\lambda) - \frac{2\alpha_8 - \beta_4 - 1}{\alpha_8 \pm 1}\, x\Big), \tag{4.57}$$

for invariant $\lambda = \ln u - (\alpha_8 \pm 1)x$.

From Proposition 4 applied to the operator Z^{23} (for $a_8 \pm 1 \neq 0$, $2\alpha_8 - \beta_4 - 1 \neq 0$) we obtain the additional operator

$$X^{(2)} = (2\alpha_8 - \beta_4)t\,\partial_t + (\alpha_8 \pm 1)\,\partial_x + u\,\partial_u. \tag{4.58}$$

One can perform the algorithm for other Z^i s of (4.47) similarly. The preliminary group classification of nonlinear fin equation (1.1) admitting an extension $\mathcal{L}_2$ of the principal Lie algebra $\mathcal{L}_1$ is listed in Table 3.

5 Conclusion

Symmetry analysis for equations $u_t = [E(x,u)u_x]_x + H(x,u)$ rather than previous results on special cases of this equation equation [19, 20], is carried out exhaustively. Also, equivalence classification is given of the equation admitting an extension by one of the principal Lie algebra of the equation. The paper is one of few applications of a new algebraic approach to the problem of group classification: the method of preliminary group classification. Derived results are summarized in Table 3.

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